

# EXISTENCE OF GROUNDSTATES FOR A CLASS OF NONLINEAR CHOQUARD EQUATIONS IN THE PLANE

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ABSTRACT. We prove the existence of a nontrivial groundstate solution for the class of nonlinear Choquard equation

$$-\Delta u + u = (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^2,$$

where  $I_\alpha$  is the Riesz potential of order  $\alpha$  on the plane  $\mathbb{R}^2$  under general nontriviality, growth and subcriticality on the nonlinearity  $F \in C^1(\mathbb{R}, \mathbb{R})$ .

## 1. INTRODUCTION

We are interested in the existence of nontrivial solutions to the class of nonlinear Choquard equations of the form

$$(\mathcal{P}) \quad -\Delta u + u = (I_\alpha * F(u))F'(u) \quad \text{in } \mathbb{R}^N,$$

where  $N \in \mathbb{N} = \{1, 2, \dots\}$ ,  $\Delta$  is the standard Laplacian operator on the Euclidean space  $\mathbb{R}^N$ ,  $I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R}$  is the Riesz potential of order  $\alpha \in (0, N)$  defined for each  $x \in \mathbb{R}^N \setminus \{0\}$  by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|x|^{N-\alpha}},$$

and a nonlinearity is described by the function  $F \in C^1(\mathbb{R}, \mathbb{R})$ . Solutions of the equation  $(\mathcal{P})$  are at least formally critical points of the energy functional defined for a function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$  by

$$(1) \quad \mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u).$$

In the particular case where for each  $s \in \mathbb{R}$ ,  $F(s) = s^2/2$ , solutions to the Choquard equation  $(\mathcal{P})$  are standing waves solutions of the Hartree equation. In particular when  $N = 3$  and  $\alpha = 2$ , the problem  $(\mathcal{P})$  has arisen in various fields of physics: quantum mechanics [20], one-component plasma [11] and self-gravitating matter [15]. In these cases, many existence results have been obtained in literature, with both variational [11, 13, 14] and ordinary differential equations techniques [6, 15, 21] (see also the review [18]). Such methods extend also to the case of homogeneous nonlinearities [16].

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When the nonlinearity  $F$  is not any more homogeneous, it has been shown that the Choquard equation  $(\mathcal{P})$  has a nontrivial solution if the nonlinearity  $F$  satisfies the following hypotheses [17]:

- $(F'_0)$  there exists  $s_0 \in \mathbb{R}$  such that  $F(s_0) \neq 0$ ;
- $(F'_1)$  there exists  $C > 0$  such that  $|F'(s)| \leq C(|s|^{\frac{\alpha}{N}} + |s|^{\frac{\alpha+2}{N-2}})$  for every  $s > 0$ ;
- $(F'_2)$   $\lim_{s \rightarrow 0} F(s)/|s|^{1+\frac{\alpha}{N}} = 0 = \lim_{s \rightarrow 0} F(s)/|s|^{\frac{N+\alpha}{N-2}}$ .

The solution  $u$  is a *groundstate*, in the sense that  $u$  minimizes the value of the functional  $\mathcal{I}$  among all nontrivial solutions. The assumptions  $(F'_0)$ ,  $(F'_1)$  and  $(F'_2)$  are rather mild and reasonable and are “almost necessary” in the sense of Berestycki and Lions [3]: the nontriviality of the nonlinearity condition  $(F'_0)$  is clearly necessary to have a nontrivial solution; the assumption  $(F'_1)$  secures a proper variational formulation of the problem  $(\mathcal{P})$  by ensuring that the energy functional  $\mathcal{I}$  is well-defined on the natural Sobolev space  $H^1(\mathbb{R}^N)$  through the Hardy–Littlewood–Sobolev and Sobolev inequalities; the condition  $(F'_2)$  is a sort of *subcriticality* condition with respect to the limiting-case embeddings. The analysis by a Pohožaev identity shows that the assumptions  $(F'_1)$  and  $(F'_2)$  are necessary in the homogeneous case  $F(s) = s^p/p$  [16].

The results in [17] can thus be seen as a counterpart for Choquard-type equations of the result of Berestycki and Lions [3] which give similar “almost necessary” conditions for the existence of a groundstate to the equation

$$(2) \quad -\Delta u + u = G'(u) \quad \text{in } \mathbb{R}^N.$$

The latter equation can be at least formally be obtained by  $(\mathcal{P})$  by passing to the limit as  $\alpha \rightarrow 0$  and setting  $G = F^2/2$ .

Whereas the above-mentioned almost necessary conditions for existence of the Choquard equation  $(\mathcal{P})$  and for the scalar field equation (2) have been obtained in higher dimensions  $N \geq 3$ , the latter result has been extended to the two-dimensional case [4], under the following assumptions

- $(G_0)$  there exists  $s_0 \in \mathbb{R}$  such that  $G(s_0) > \frac{|s_0|^2}{2}$ ;
- $(G_1)$  for every  $\theta > 0$  there exists  $C = C_\theta > 0$  such that  $|G'(s)| \leq C_\theta \min\{1, s^2\}e^{\theta|s|^2}$  for every  $s > 0$ ;
- $(G_2)$   $\lim_{s \rightarrow 0} G(s)/|s|^2 < 1/2$ .

This raises naturally the question whether there is a similar existence result for the Choquard equation  $(\mathcal{P})$  in the planar case.

In the present work, we provide a general existence result for groundstate solutions of problem  $(\mathcal{P})$  in the planar case  $N = 2$ , which is a two-dimensional counterpart of [17] and a counterpart for the Choquard equation of [4]. The counterparts of  $(F'_0)$ ,  $(F'_1)$ ,  $(F'_2)$  we need are the following:

- $(F_0)$  there exists  $s_0 \in \mathbb{R}$  such that  $F(s_0) \neq 0$ ;
- $(F_1)$  for every  $\theta > 0$  there exists  $C = C_\theta > 0$  such that  $|F'(s)| \leq C_\theta \min\{1, |s|^{\frac{\alpha}{2}}\}e^{\theta|s|^2}$  for every  $s > 0$ ;
- $(F_2)$   $\lim_{s \rightarrow 0} F(s)/|s|^{1+\frac{\alpha}{2}} = 0$ .

Our main result reads as follows:

**Theorem 1.1.** *If  $N = 2$  and  $F \in C^1(\mathbb{R}, \mathbb{R})$  satisfies the conditions  $(F_0)$ ,  $(F_1)$  and  $(F_2)$ , then the problem  $(\mathcal{P})$  has a groundstate solution  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ , namely the function  $u$  solves  $(\mathcal{P})$  and*

$$\mathcal{I}(u) = c := \inf \{ \mathcal{I}(v) \mid v \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ is a solution of } (\mathcal{P}) \}.$$

Let us discuss the assumptions of Theorem 1.1. As above, the assumption  $(F_0)$  is necessary for the existence of a nontrivial solution. As before, the condition  $(F_1)$  ensures needed the well-definiteness of the energy functional on the whole space  $H^1(\mathbb{R}^2)$ . It has a different shape, because in dimension  $N = 2$ , the critical nonlinearity for Sobolev embeddings is not anymore a power but rather an exponential-type nonlinearity. More precisely, the integral of  $\min\{1, u^2\}e^{\theta|u|^2}$  on  $\mathbb{R}^2$  is uniformly controlled on  $H_0^1(B_1)$  if and only if  $\theta \int_{B_1} |\nabla u|^2 \leq 4\pi$  (see [1, 19]); this is why the parameter  $\theta > 0$  appears in condition  $(F_1)$ . It will appear that the condition  $(F_1)$  is strong enough at infinity. Indeed, by integrating the function  $F'$ , it is possible to observe that for every  $\theta > 0$ ,

$$(3) \quad \lim_{|s| \rightarrow \infty} \frac{|F(s)| + |F'(s)||s|}{e^{\theta|s|^2}} = 0.$$

A subcriticality condition still needs to be imposed around 0; that is the goal of the subcriticality condition  $(F_2)$ .

The assumptions  $(F_0)$ ,  $(F_1)$  and  $(F_2)$  are still *almost necessary*: in the case  $F(s) = \frac{s^p}{p}$ , they are satisfied if and only if  $p > 1 + \frac{\alpha}{2}$ , and for  $p \leq 1 + \frac{\alpha}{2}$  the Choquard equation  $(\mathcal{P})$  has no nontrivial solutions (see [16]).

In order to prove Theorem 1.1 the constraint minimization technique used in [3, 4] for the local problem (2) does not seem to work, as it introduces a Lagrange multiplier that cannot be absorbed through a suitable dilation because of the presence of three different scalings in the equation and of the nonhomogeneity of the nonlinearity.

Following [17], we use a mountain-pass construction. We start by constructing a Palais–Smale sequence for the mountain-pass level

$$(4) \quad b := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(\gamma(t)),$$

where

$$(5) \quad \Gamma := \{ \gamma \in C([0,1], H^1(\mathbb{R}^2)) \mid \gamma(0) = 0 \text{ and } \mathcal{I}(\gamma(1)) < 0 \}.$$

To avoid relying on an Ambrosetti–Rabinowitz superlinearity condition, we use a scaling trick due to Jeanjean [9], which allows to construct Pohožaev–Palais–Smale sequence (Proposition 3.1), namely a Palais–Smale sequence which, in addition, satisfies asymptotically the Pohožaev identity

$$(6) \quad \mathcal{P}(u) := \int_{\mathbb{R}^2} |u|^2 - \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u)) F(u) = 0.$$

Such a condition will imply quite directly the boundedness of the sequence in the space  $H^1(\mathbb{R}^2)$  and it will be crucial to get the convergence, hence the existence of a solution (Proposition 4.1).

We are left with showing that the solution  $u$  is actually a groundstate. To prove this, we first show that the solution  $u$  itself satisfies the Pohožaev identity (Proposition 5.2).

This will follow by simple calculations once a suitable regularity result is established (Proposition 5.1); this regularity turns out to be easier to prove from the assumption  $(F_1)$  than in the higher-dimensional case [17] where a suitable nonlocal Brezis–Kato regularity had to be proved. The last ingredient that we need is an optimal path  $\gamma_v \in \Gamma$  associated to any solution  $v$  of  $(\mathcal{P})$ . The construction of such paths (Proposition 5.3) is inspired by [10, 17] but it is more delicate in our two-dimensional case than in the higher dimensions  $N \geq 3$ , because dilations  $t \mapsto v(\cdot/t) \in H^1(\mathbb{R}^N)$  are not anymore continuous at  $t = 0$  when  $N = 2$ .

The content of the paper is the following: in Section 2 we provide some technical preliminaries; in Section 3 we construct the Pohožaev–Palais–Smale sequence; in Section 4 we show that the sequence converges to a solution of  $(\mathcal{P})$ ; in Section 5 we prove that  $u$  is actually a groundstate. In the last section we also state some qualitative result concerning the solutions, which can be proved directly following [17].

## 2. PRELIMINARIES

In this section we present some preliminary results which we will need throughout the rest of this paper. We start by reformulating in a more convenient form the Moser–Trudinger inequality of Adachi and Tanaka [1]. This quantitative estimate will play a crucial role throughout the paper.

**Proposition 2.1** (Moser–Trudinger inequality). *For any  $\beta \in (0, 4\pi)$  there exists  $C = C_\beta > 0$  such that for every  $u \in H^1(\mathbb{R}^2)$  satisfying*

$$\int_{\mathbb{R}^2} |\nabla u|^2 \leq 1,$$

*one has*

$$\int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\beta|u|^2} \leq C_\beta \int_{\mathbb{R}^2} |u|^2$$

*Proof.* The result follows the fact [1, Theorem 0.1] that under the conditions of the theorem,

$$\int_{\mathbb{R}^2} (e^{\beta|u|^2} - 1) \leq C \int_{\mathbb{R}^2} |u|^2.$$

together with the elementary inequalities valid for every  $s \geq 0$ ,

$$\left(1 - \frac{1}{e}\right) \max\{1, s\} e^s \leq e^s - 1 \leq \max\{1, s\} e^s. \quad \square$$

We will also use the Hardy–Littlewood–Sobolev inequality to deal with the nonlocal term (see for example [12, Theorem 4.3]):

**Proposition 2.2** (Hardy–Littlewood–Sobolev inequality). *For any  $p \in [1, \frac{2}{\alpha})$  and  $f \in L^p(\mathbb{R}^2)$  there exists a constant  $C = C_{\alpha,p}$  such that*

$$\|I_\alpha * f\|_{L^{\frac{2p}{2-\alpha p}}(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}.$$

Combining the last two results with the assumption on  $F$  and (3) we deduce that the energy functional is well-defined on  $H^1(\mathbb{R}^2)$ :

**Proposition 2.3.** *If  $F$  satisfies  $(F_1)$ , then the energy functional  $\mathcal{I}$  defined by (1) is well-defined and continuously differentiable.*

*Proof.* We first consider the superposition map  $\mathcal{E}$  defined for each  $u \in H^1(\mathbb{R}^2)$  and  $x \in \mathbb{R}^2$  by  $\mathcal{E}(u)(x) = F'(u(x))$ . We claim that  $\mathcal{E}$  is well-defined and continuous as a map from  $H^1(\mathbb{R}^2)$  to  $L^{4/\alpha}(\mathbb{R}^2)$ . Indeed by assumption  $(F_1)$ , for every  $\theta > 0$ , and  $s \in \mathbb{R}$ , we have

$$|F'(s)|^{\frac{4}{\alpha}} \leq C_{\theta}^{\frac{4}{\alpha}} \min\{1, s^2\} e^{\frac{4\theta}{\alpha}|s|^2}.$$

If  $u \in H^1(\mathbb{R}^2)$ , we take  $\theta > 0$  such that  $\int_{\mathbb{R}^2} |\nabla u|^2 < \frac{\alpha\pi}{2\theta}$ . We observe that

$$|F'(u)|^{\frac{4}{\alpha}} \leq C_{\theta}^{\frac{4}{\alpha}} \min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2}$$

on  $\mathbb{R}^2$ , where the right-hand side is integrable in view of the Moser–Trudinger inequality (Proposition 2.1); therefore the map  $\mathcal{E} : H^1(\mathbb{R}^2) \rightarrow L^{4/\alpha}(\mathbb{R}^2)$  is well-defined.

If now the sequence  $(u_n)_{n \in \mathbb{N}}$  converges to  $u$  in  $H^1(\mathbb{R}^2)$ , then we can assume without loss of generality that  $\nu := \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} |\nabla u_n|^2 < \frac{\alpha\pi}{2\theta}$  and that  $(u_n)_{n \in \mathbb{N}}$  converges to  $u$  almost everywhere. We have then for some constant  $C \geq 0$ ,

$$C(\min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2} + \min\{1, |u_n|^2\} e^{\frac{4\theta}{\alpha}|u_n|^2}) - |F'(u) - F'(u_n)|^{\frac{4}{\alpha}} \geq 0,$$

for each  $n \in \mathbb{N}$  almost everywhere in  $\mathbb{R}^2$ . By Fatou's lemma we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} C(\min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2} + \min\{1, |u_n|^2\} e^{\frac{4\theta}{\alpha}|u_n|^2}) - |F'(u) - F'(u_n)|^{\frac{4}{\alpha}} \\ \geq 2C \int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2} \end{aligned}$$

and therefore

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^2} |F'(u) - F'(u_n)|^{\frac{4}{\alpha}} \leq C \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^2} \min\{1, |u_n|^2\} e^{\frac{4\theta}{\alpha}|u_n|^2} - \min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2}.$$

If we consider the set  $A_n^\lambda = \{x \in \mathbb{R}^2 \mid |u_n(x)| \geq \lambda\}$ , we have by Lebesgue's dominated convergence theorem, for every  $\lambda > 0$ ,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^2 \setminus A_n^\lambda} \min\{1, |u_n|^2\} e^{\frac{4\theta}{\alpha}|u_n|^2} \\ \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^2 \setminus A_n^\lambda} (\min\{1, |u_n|^2\} e^{\frac{4\theta}{\alpha}|u_n|^2} - \min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2}) \\ + \int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2} \\ \leq \int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{\frac{4\theta}{\alpha}|u|^2}. \end{aligned}$$

On the other hand, we have by the Cauchy–Schwarz inequality, the Chebyshev inequality and the Moser–Trudinger inequality (Proposition 2.1)

$$\int_{A_n^\lambda} \min\{1, |u_n|^2\} e^{\frac{4\theta}{\alpha}|u_n|^2} \leq |A_n^\lambda|^{\frac{1}{2}} \left( \int_{A_n^\lambda} \min\{1, |u_n|^2\} e^{\frac{8\theta}{\alpha}|u_n|^2} \right)^{\frac{1}{2}} \leq \frac{C}{\lambda} \int_{\mathbb{R}^2} |u_n|^2.$$

This allows to conclude that the map  $\mathcal{E} : H^1(\mathbb{R}^2) \rightarrow L^{4/\alpha}(\mathbb{R}^2)$  is continuous.

We now consider the map  $\mathcal{F} : H^1(\mathbb{R}^2) \rightarrow L^{4/(2+\alpha)}(\mathbb{R}^2)$  defined for each  $u \in H^1(\mathbb{R}^2)$  by  $\mathcal{F}(u) = F \circ u$ . We observe that for every  $s \in \mathbb{R}$ ,

$$F(s) = \int_0^1 F'(\tau s) s \, d\tau,$$

and thus for almost every  $x \in \mathbb{R}^2$ ,

$$F(u(x)) = \int_0^1 F'(\tau u(x)) u(x) \, d\tau.$$

It follows thus from the first part of the proof that  $\mathcal{F}$  is well-defined from  $H^1(\mathbb{R}^2)$  to  $L^{4/(2+\alpha)}(\mathbb{R}^2)$ .

For the differentiability we consider a sequence  $(u_n)_{n \in \mathbb{N}}$  converging strongly to  $u$  in  $H^1(\mathbb{R}^2)$ . We observe that for each  $n \in \mathbb{N}$ ,

$$\mathcal{F}(u_n) - \mathcal{F}(u) - \mathcal{E}(u)(u_n - u) = \int_0^1 (\mathcal{E}((1 - \tau)u + \tau(u_n)) - \mathcal{E}(u))(u_n - u) \, d\tau,$$

and thus by Hölder's inequality

$$\|\mathcal{F}(u_n) - \mathcal{F}(u) - \mathcal{E}(u)(u_n - u)\|_{L^{4/(2+\alpha)}} \leq \int_0^1 \|\mathcal{E}((1 - \tau)u + \tau(u_n)) - \mathcal{E}(u)\|_{L^{4/\alpha}} \|u_n - u\|_{L^2} \, d\tau.$$

By the convergence of the sequence  $(u_n)_{n \in \mathbb{N}}$  and the continuity of the functional  $\mathcal{E}$ , it follows that, as  $n \rightarrow \infty$ ,

$$\|\mathcal{F}(u_n) - \mathcal{F}(u) - \mathcal{E}(u)(u_n - u)\|_{L^{4/(2+\alpha)}} = o(\|u_n - u\|_{L^2}),$$

that is,  $\mathcal{E}$  represents the Fréchet differential of the functional  $\mathcal{F}$ . Since  $\mathcal{E}$  is continuous, it follows that  $\mathcal{F}$  is of class  $C^1$ .

Finally, we consider the quadratic form  $\mathcal{Q}$  defined for  $f \in L^{4/(2+\alpha)}$  by

$$\mathcal{Q}(f) = \int_{\mathbb{R}^2} (I_\alpha * f) f.$$

By the Hardy–Littlewood–Sobolev inequality (Proposition 2.2), the quadratic form  $\mathcal{Q}$  is bounded on bounded sets of the space  $L^{4/(2+\alpha)}(\mathbb{R}^2)$ . This implies that  $\mathcal{Q}$  is continuously differentiable and thus the functional

$$u \in H^1(\mathbb{R}^2) \mapsto \mathcal{Q}(\mathcal{F}(u), \mathcal{F}(u)) = \int_{\mathbb{R}^2} (I_\alpha * F(u)) F(u)$$

is continuously differentiable. By the smoothness of the norm on a Hilbert space, we conclude that the functional  $\mathcal{I}$  is continuously differentiable.  $\square$

Finally, we will use the following improvement of Proposition 2.2 when one has some more  $L^p$  integrability:

**Proposition 2.4.** *For any  $p \in [1, \frac{2}{\alpha})$ ,  $q \in (\frac{2}{\alpha}, +\infty)$  and  $f \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$  there exists  $C = C_{\alpha,p,q}$  such that*

$$\|I_\alpha * f\|_{L^\infty(\mathbb{R}^2)} \leq C(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^q(\mathbb{R}^2)}).$$

*Proof.* The result is classical. We give its short proof for the convenience of the reader. By choosing  $p, q$  in those range we have  $(2 - \alpha)\frac{q}{q-1} < 2 < (2 - \alpha)\frac{p}{p-1}$ ; therefore, through splitting the integral and Hölder inequality we get for every  $x \in \mathbb{R}^2$

$$\begin{aligned} |I_\alpha * f(x)| &\leq C \int_{\mathbb{R}^2} \frac{|f(x-y)|}{|y|^{2-\alpha}} dy \\ &\leq C \left( \int_{B_1} \frac{dy}{|y|^{(2-\alpha)\frac{q}{q-1}}} \right)^{1-\frac{1}{q}} \|f\|_{L^q(B_1(x))} \\ &\quad + C \left( \int_{B_1^c} \frac{dy}{|y|^{(2-\alpha)\frac{p}{p-1}}} \right)^{1-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^2 \setminus B_1(x))} \\ &\leq C' (\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^q(\mathbb{R}^2)}). \end{aligned} \quad \square$$

### 3. CONSTRUCTION OF A POHOŽAEV–PALAIS–SMALE SEQUENCE

In this section we show the existence of a Pohožev–Palais–Smale sequence at the level  $b$  defined by (4). In other words, we construct a sequence of almost critical points which asymptotically satisfies the equation  $(\mathcal{P})$  and the Pohožev identity (6).

**Proposition 3.1.** *If the function  $F \in C^1(\mathbb{R}, \mathbb{R})$  satisfies the assumptions  $(F_0)$  and  $(F_1)$ , then there exists a sequence  $(u_n)_{n \in \mathbb{N}}$  in  $H^1(\mathbb{R}^2)$  such that:*

- (a)  $\mathcal{I}(u_n) \xrightarrow{n \rightarrow \infty} b$ ;
- (b)  $\mathcal{I}'(u_n) \xrightarrow{n \rightarrow \infty} 0$  strongly in  $H^1(\mathbb{R}^2)'$ ;
- (c)  $\mathcal{P}(u_n) \xrightarrow{n \rightarrow \infty} 0$ .

To prove Proposition 3.1, we first need to show that the energy functional  $\mathcal{I}$  has the mountain pass geometry, namely that the mountain pass level  $b$  is well-defined and nontrivial:

**Lemma 3.2.** *The critical level  $b$  defined by (4) satisfies  $b \in (0, +\infty)$ .*

*Proof.* We start by showing the finiteness of  $b$ , which will be done as in [17, Proposition 2.1]. By the definition of the set  $b$ , it is sufficient to show that  $\Gamma \neq \emptyset$ , which in turn is equivalent to find  $u_0 \in H^1(\mathbb{R}^2)$  such that  $\mathcal{I}(u_0) < 0$ . By the assumption  $(F_0)$ , we can take  $s_0$  such that  $F(s_0) \neq 0$  and we find

$$\int_{\mathbb{R}^2} (I_\alpha * F(s_0 \mathbf{1}_{B_1})) F(s_0 \mathbf{1}_{B_1}) = F(s_0)^2 \int_{B_1} \int_{B_1} I_\alpha(x-y) dx dy > 0;$$

therefore by density of smooth functions in  $L^q(\mathbb{R}^2)$  there will be  $v_0 \in H^1(\mathbb{R}^2)$  with  $\int_{\mathbb{R}^2} (I_\alpha * F(v_0)) F(v_0) > 0$ . We consider now, for  $t > 0$ , the function  $v_t : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined for  $x \in \mathbb{R}^2$  by  $v_t(x) := v_0(\frac{x}{t})$ . This function verifies

$$\mathcal{I}(v_t) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v_0|^2 + \frac{t^2}{2} \int_{\mathbb{R}^2} |v_0|^2 - \frac{t^{2+\alpha}}{2} \int_{\mathbb{R}^2} (I_\alpha * F(v_0)) F(v_0),$$

therefore, for some  $t_0 \gg 0$ , the function  $u_0 := v_{t_0}$  satisfies  $\mathcal{I}(u_0) < 0$ .

Let us now show that  $b > 0$ . By the definition of  $b$ , it is equivalent to show that there exists  $\varepsilon > 0$  such that for every path  $\gamma \in \Gamma$  there exists  $t_\gamma \in [0, 1]$  with  $\mathcal{I}(\gamma(t_\gamma)) \geq \varepsilon > 0$ .

We first assume that  $u \in H^1(\mathbb{R}^2)$  and  $\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) \leq \delta \ll 1$ . In particular, since  $\int_{\mathbb{R}^2} |\nabla u|^2 \leq 1$ , Proposition 2.1 applies to  $u$  with  $\beta = 2\pi$ . Therefore, by Propositions 2.2, and 2.1 and by (3), we have

$$\begin{aligned} \int_{\mathbb{R}^2} (I_\alpha * F(u))F(u) &\leq C \left( \int_{\mathbb{R}^2} |F(u)|^{\frac{4}{2+\alpha}} \right)^{1+\frac{\alpha}{2}} \leq C \left( \int_{\mathbb{R}^2} \min\{1, |u|^2\} e^{2\pi|u|^2} \right)^{1+\frac{\alpha}{2}} \\ &\leq C \left( \int_{\mathbb{R}^2} |u|^2 \right)^{1+\frac{\alpha}{2}}, \end{aligned}$$

which is smaller than  $\frac{1}{4} \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2)$  if  $\delta$  is small enough. It follows then that if  $\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) \leq \delta$ , we have

$$\mathcal{I}(u) \geq \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2).$$

We now take an arbitrary path  $\gamma \in \Gamma$ . Since  $\mathcal{I}(\gamma(1)) < 0 < \frac{1}{4} \int_{\mathbb{R}^2} (|\nabla \gamma(t_\gamma)|^2 + |\gamma(t_\gamma)|^2)$ , we have

$$\int_{\mathbb{R}^2} (|\nabla \gamma(1)|^2 + |\gamma(1)|^2) > \delta > 0 = \int_{\mathbb{R}^2} (|\nabla \gamma(0)|^2 + |\gamma(0)|^2);$$

therefore, there exists  $t_\gamma \in (0, 1)$  such that  $\int_{\mathbb{R}^2} (|\nabla \gamma(t_\gamma)|^2 + |\gamma(t_\gamma)|^2) = \delta$ , and hence  $\mathcal{I}(\gamma(t_\gamma)) \geq \frac{\delta}{4}$ . The lemma follows by taking  $\varepsilon := \frac{\delta}{4}$ .  $\square$

*Proof of Proposition 3.1.* We follow [8, Chapter 4; 9, Chapter 2; 17, Proposition 2.1]. We consider the map  $\Phi$  given by

$$\begin{aligned} \Phi : \mathbb{R} \times H^1(\mathbb{R}^2) &\longrightarrow H^1(\mathbb{R}^2) \\ (\sigma, v) &\longmapsto \Phi(\sigma, v)(x) := v(e^{-\sigma}x) \end{aligned}$$

and the functional  $\tilde{\mathcal{I}} = \mathcal{I} \circ \Phi$ :

$$\tilde{\mathcal{I}}(\sigma, v) = \mathcal{I}(\Phi(\sigma, v)) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{e^{2\sigma}}{2} \int_{\mathbb{R}^2} |v|^2 - \frac{e^{(2+\alpha)\sigma}}{2} \int_{\mathbb{R}^2} (I_\alpha * F(v))F(v),$$

which is well-defined and Fréchet-differentiable on the Hilbert space  $\mathbb{R} \times H^1(\mathbb{R}^2)$ .

We define now the class of paths

$$\tilde{\Gamma} := \left\{ \tilde{\gamma} \in C([0, 1], \mathbb{R} \times H^1(\mathbb{R}^2)) \mid \tilde{\gamma}(0) = (0, 0) \text{ and } \tilde{\mathcal{I}}(\tilde{\gamma}(1)) < 0 \right\};$$

since we have  $\Gamma = \{\Phi \circ \tilde{\gamma} \mid \tilde{\gamma} \in \tilde{\Gamma}\}$ , the mountain pass levels of  $\mathcal{I}$  and  $\tilde{\mathcal{I}}$  coincide, namely

$$b = \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{t \in [0, 1]} \tilde{\mathcal{I}}(\tilde{\gamma}(t)).$$

Since, by Lemma 3.2, the mountain pass level  $b$  is not trivial, we can thus apply the minimax principle ([24], Theorem 2.9) and we find a sequence  $((\sigma_n, v_n))_{n \in \mathbb{N}}$  in  $\mathbb{R} \times H^1(\mathbb{R}^2)$  such that:

$$\tilde{\mathcal{I}}(\sigma_n, v_n) \xrightarrow{n \rightarrow \infty} b \quad \text{and} \quad \tilde{\mathcal{I}}(\sigma_n, v_n) \xrightarrow{n \rightarrow \infty} 0 \text{ strongly in } (\mathbb{R} \times H^1(\mathbb{R}^2))'.$$

By writing explicitly the derivative of  $\tilde{\mathcal{I}}$ :

$$\tilde{\mathcal{I}}'(\sigma_n, v_n)[h, w] = \mathcal{I}'(\Phi(\sigma_n, v_n))[\Phi(\sigma_n, w)] + \mathcal{P}(\Phi(\sigma_n, v_n))h;$$

we see that the conclusion follows by taking  $u_n = \Phi(\sigma_n, v_n)$ .  $\square$



## 4. CONVERGENCE OF THE POHOŽAEV–PALAIS–SMALE SEQUENCE

In this Section we will construct a nontrivial solution of  $(\mathcal{P})$  from the sequence given by Proposition 3.1.

**Proposition 4.1.** *If the function  $F \in C^1(\mathbb{R}, \mathbb{R})$  satisfies  $(F_1)$  and  $(F_2)$  and the sequence  $(u_n)_{n \in \mathbb{N}}$  in  $H^1(\mathbb{R}^2)$  satisfies*

- (a)  $\mathcal{I}(u_n)$  is uniformly bounded,
- (b)  $\mathcal{I}'(u_n) \xrightarrow{n \rightarrow \infty} 0$  strongly in  $(H^1(\mathbb{R}^2))'$ ,
- (c)  $\mathcal{P}(u_n) \xrightarrow{n \rightarrow \infty} 0$ ;

*then, up to subsequences, one of the following occurs:*

- either  $u_n \xrightarrow{n \rightarrow \infty} 0$  strongly in  $H^1(\mathbb{R}^2)$ ;
- or there exists  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$  solving  $(\mathcal{P})$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^2$  such that  $u_n(\cdot - x_n) \rightharpoonup u$  weakly in  $H^1(\mathbb{R}^2)$ .

We follow the strategy of [17, Proposition 2.2]. Since the gradient does not appear in the Pohožaev identity (6), it will be more delicate to show that the nonlocal term does not vanish.

*Proof of Proposition 4.1.* We assume that the first alternative does not hold, namely

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2) > 0.$$

By writing for each  $n \in \mathbb{N}$

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{\alpha}{2(2+\alpha)} \int_{\mathbb{R}^2} |u_n|^2 = \mathcal{I}(u_n) - \frac{\mathcal{P}(u_n)}{2+\alpha}$$

we deduce that the sequence  $(u_n)_{n \in \mathbb{N}}$  is bounded in the space  $H^1(\mathbb{R}^2)$ . Since  $\mathcal{I}'(u_n) \rightarrow 0$  in  $H^1(\mathbb{R}^2)'$  as  $n \rightarrow \infty$ , we have  $\mathcal{I}'(u_n)[u_n] \rightarrow 0$  as  $n \rightarrow \infty$ , therefore

$$\int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F'(u_n) u_n = \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2) - \mathcal{I}'(u_n)[u_n] \geq \frac{1}{C}.$$

Taking  $C_0 \geq \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2)$ , we can apply Proposition 2.1 to  $\frac{1}{\sqrt{C_0}} u_n$  with  $\beta = 2\pi$  and we obtain for each  $n \in \mathbb{N}$

$$\int_{\mathbb{R}^2} \min\{1, u_n^2\} e^{\frac{2\pi}{C_0} |u_n|^2} \leq C_{2\pi} \frac{\int_{\mathbb{R}^2} |u_n|^2}{C_0} \leq C_{2\pi};$$

moreover, we also have, as  $n \rightarrow \infty$ ,

$$\begin{aligned} \int_{\mathbb{R}^2} |u_n|^2 &= \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) + \mathcal{P}(u_n) \\ &= \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) + o(1). \end{aligned}$$

Therefore, from Proposition 2.2 and by (3) we get

$$\begin{aligned}
(7) \quad \frac{1}{C} &\leq \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F'(u_n) u_n \leq C \left( \int_{\mathbb{R}^2} |F(u_n)|^{\frac{4}{2+\alpha}} \int_{\mathbb{R}^2} (|F'(u_n)| |u_n|)^{\frac{4}{2+\alpha}} \right)^{\frac{2+\alpha}{4}} \\
&\leq C' \left( \int_{\mathbb{R}^2} \min\{1, |u_n|^2\} e^{\frac{2\pi}{C_0} |u_n|^2} \right)^{1+\frac{\alpha}{2}} \leq C'' \left( \int_{\mathbb{R}^2} |u_n|^2 \right)^{1+\frac{\alpha}{2}} \\
&= C'' \left( \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) + o(1) \right)^{1+\frac{\alpha}{2}},
\end{aligned}$$

namely  $\int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n)$  is bounded from above from zero when  $n \rightarrow \infty$ .

We now want to prove that  $u_n$  does not vanish. We will use the following inequality [13, Lemma I.1] (see also [16, lemma 2.3; 22, (2.4); 24, lemma 1.21]):

$$\int_{\mathbb{R}^2} |u_n|^p \leq C \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2) \left( \sup_{x \in \mathbb{R}^2} \int_{B_1(x)} |u_n|^p \right)^{1-\frac{2}{p}}$$

and we will show that the right-hand side term is bounded from below by a positive constant, for every  $p > 2$ . By the assumption  $(F_2)$  and (3), for every  $\varepsilon > 0$  there exists  $C_{\varepsilon, \theta} > 0$  such that

$$|F(s)|^{\frac{4}{2+\alpha}} \leq \varepsilon \min\{1, |s|^2\} e^{\theta |s|^2} + C_{\varepsilon, \theta} |s|^p;$$

therefore

$$\begin{aligned}
(8) \quad \left( \sup_{x \in \mathbb{R}^2} \int_{B_1(x)} |u_n|^p \right)^{1-\frac{2}{p}} &\geq \frac{1}{C} \frac{\int_{\mathbb{R}^2} |u_n|^p}{\int_{\mathbb{R}^2} (|\nabla u_n|^2 + |u_n|^2)} \\
&\geq \frac{1}{CC_0 C_\varepsilon} \left( \int_{\mathbb{R}^2} |F(u_n)|^{\frac{4}{2+\alpha}} - \varepsilon \int_{\mathbb{R}^2} \min\{1, |u_n|^2\} e^{\frac{2\pi}{C_0} |u_n|^2} \right) \\
&\geq \frac{1}{C'_\varepsilon} \left( \left( \int_{\mathbb{R}^2} (I_\alpha * F(u_n)) F(u_n) \right)^{\frac{2}{2+\alpha}} - \varepsilon C \int_{\mathbb{R}^2} |u_n|^2 \right) \\
&\geq \frac{1}{C'_\varepsilon} \left( \frac{1}{C'} - \varepsilon CC_0 \right).
\end{aligned}$$

The quantity  $\varepsilon$  being arbitrary, we get  $\int_{B_1(x_n)} |u_n|^p \geq \frac{1}{C}$  for some  $x_n \in \mathbb{R}^2$ , for  $n$  large enough.

We can now consider the translated sequence  $(u_n(\cdot - x_n))_{n \in \mathbb{N}}$ . Since the problem  $(\mathcal{P})$  is invariant by translation, this sequence will satisfy the hypotheses of the present proposition, hence we will still denote it as  $(u_n)_{n \in \mathbb{N}}$  and we will assume that  $x_n = 0$  for all  $n \in \mathbb{N}$ . Since  $\liminf_{n \rightarrow \infty} \int_{B_1} |u_n|^p > 0$ , we can assume that this sequence  $(u_n)_{n \in \mathbb{N}}$  converges weakly to  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$ . We just have to show that  $u$  solves  $(\mathcal{P})$ .

The sequence  $(u_n)_{n \in \mathbb{N}}$  being bounded in  $H^1(\mathbb{R}^2)$ , the sequence  $(F(u_n))_{n \in \mathbb{N}}$  is bounded in  $L^p(\mathbb{R}^2)$  for every  $p \geq \frac{4}{2+\alpha}$ . Moreover, up to subsequences,  $u_n \rightarrow u$  almost everywhere as  $n \rightarrow \infty$ , so by the continuity of the function  $F$  we also have  $F(u_n) \rightarrow F(u)$  almost everywhere as  $n \rightarrow \infty$ ; this implies that  $F(u_n) \rightharpoonup F(u)$  weakly in  $L^p(\mathbb{R}^2)$  for every such  $p$  as  $n \rightarrow \infty$ . Since  $\frac{2}{\alpha} > \frac{4}{2+\alpha}$ , by Propositions 2.2 and 2.4 we get  $I_\alpha * F(u_n) \rightharpoonup I_\alpha * F(u)$

weakly in  $L^{4/(2-\alpha)}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$  as  $n \rightarrow \infty$ . By the condition  $(F_1)$  and Proposition 2.1, the sequence  $(F'(u_n))_{n \in \mathbb{N}}$  is bounded in  $L^p(\mathbb{R}^2)$  for every  $p \in [\frac{2}{\alpha}, +\infty)$ , and by continuity  $F'(u_n) \rightarrow F'(u)$  almost everywhere as  $n \rightarrow \infty$ ; therefore,  $F'(u_n) \rightarrow F'(u)$  strongly in  $L^q_{\text{loc}}(\mathbb{R}^2)$  for every  $q \in [1, +\infty)$  as  $n \rightarrow \infty$ , hence

$$(I_\alpha * F(u_n))F'(u_n) \xrightarrow{n \rightarrow \infty} (I_\alpha * F(u))F'(u) \quad \text{in } L^r_{\text{loc}}(\mathbb{R}^2) \quad \forall r \in [1, +\infty).$$

Therefore, for every  $\varphi \in C_0^1(\mathbb{R}^2)$ ,

$$\begin{aligned} (9) \quad \int_{\mathbb{R}^2} (\nabla u \cdot \nabla \varphi + u\varphi) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} (\nabla u_n \cdot \nabla \varphi + u_n \varphi) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} (I_\alpha * F(u_n))F'(u_n)\varphi = \int_{\mathbb{R}^2} (I_\alpha * F(u))F'(u)\varphi, \end{aligned}$$

namely  $u$  solves the Choquard equation  $(\mathcal{P})$ .  $\square$

**Corollary 4.2.** *If  $F$  satisfies the conditions  $(F_0)$ ,  $(F_1)$  and  $(F_2)$ , then problem  $(\mathcal{P})$  has a nontrivial solution  $u \in H^1(\mathbb{R}^2)$ .*

*Proof.* By Proposition 3.1,  $\mathcal{I}$  admits a Pohožaev–Palais–Smale sequence  $(u_n)_{n \in \mathbb{N}}$  at the level  $b$ . We apply Proposition 4.1 to  $(u_n)_{n \in \mathbb{N}}$ . If the first alternative occurred, then we would have  $\mathcal{I}(u_n) \rightarrow \mathcal{I}(0) = 0$  as  $n \rightarrow \infty$ , in contradiction with Lemma 3.2. Therefore, the second alternative must occur, and in particular we get a solution  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$  of  $(\mathcal{P})$ .  $\square$

## 5. FROM SOLUTIONS TO GROUNDSTATES

We start by providing a local regularity result for solution of  $(\mathcal{P})$ . This result can be obtained quite directly because our growth assumption  $(F_1)$  gives a good control on  $I_\alpha * F(u)$  which, in turn, permits to apply a standard bootstrap method. The equivalent result in higher dimension  $N \geq 3$  is more delicate to prove (see [17, Theorem 2]) because of the relative weakness of assumption  $(F'_1)$ .

**Proposition 5.1.** *If  $F \in C^1(\mathbb{R}, \mathbb{R})$  satisfies the condition  $(F_1)$  and if the function  $u \in H^1(\mathbb{R}^2)$  solves the problem  $(\mathcal{P})$ , then  $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$  for every  $p \geq 1$ .*

*Proof.* By (3) and Lemma 2.1 we deduce that if  $v \in H^1(\mathbb{R}^2)$  then  $F(v) \in L^p(\mathbb{R}^2)$  for every  $p \geq \frac{4}{2+\alpha}$ . Since  $\frac{2}{\alpha} > \frac{4}{2+\alpha}$ , by Proposition 2.4 inequality we get  $I_\alpha * F(v) \in L^\infty(\mathbb{R}^2)$ . Therefore, any solution  $u$  of  $(\mathcal{P})$  verifies

$$|-\Delta u + u| \leq C|F'(u)|,$$

with  $F'(u) \in L^p_{\text{loc}}(\mathbb{R}^2)$  for every  $p \geq 1$  because of  $(F_1)$ . By standard (interior) regularity theory on bounded domains (see for example [7, Chapter 9]) we deduce that  $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^2)$ .  $\square$

The extra regularity just proved allows to prove that solutions of  $(\mathcal{P})$  satisfy the Pohožaev identity (6). The proof of the Pohožaev identity is classical and it is based on testing  $(\mathcal{P})$  against a suitable cut-off of  $x \cdot \nabla u(x)$ , therefore it will be skipped. Details can be found in [17, Theorem 3].

**Proposition 5.2** (Pohožaev identity). *If  $F \in C^1(\mathbb{R}, \mathbb{R})$  satisfies  $(F_1)$  and  $u \in H^1(\mathbb{R}^2) \cap W_{\text{loc}}^{2,2}(\mathbb{R}^2)$  solves  $(\mathcal{P})$ , then*

$$\mathcal{P}(u) = \int_{\mathbb{R}^2} |u|^2 - \left(1 + \frac{\alpha}{2}\right) \int_{\mathbb{R}^2} (I_\alpha * F(u)) F(u) = 0.$$

The Pohožaev identity allows us to show that the mountain pass solution is actually a groundstate. We will argue like [10, Lemma 2.1; 17, Proposition 2.1], associating to any solution  $v$  a path  $\gamma_v \in \Gamma$  passing through  $v$ . The main difficulty here is that the integral of  $|\nabla u|^2$  is invariant by dilation, therefore we are not allowed to join  $v$  with 0 by just taking dilations  $t \mapsto v(\frac{\cdot}{t})$ . To overcome this difficulty, we will combine properly dilations and multiplication by constants [10].

**Proposition 5.3.** *If  $F \in C^1(\mathbb{R}, \mathbb{R})$  satisfies  $(F_1)$  and  $v \in H^1(\mathbb{R}^2) \setminus \{0\}$  solves  $(\mathcal{P})$ , then there exists a path  $\gamma_v \in C([0, 1], H^1(\mathbb{R}^2))$  such that:*

- (a)  $\gamma_v(0) = 0$ ;
- (b)  $\gamma_v(1/2) = v$ ;
- (c)  $\mathcal{I}(\gamma_v(t)) < \mathcal{I}(v)$  for every  $t \in [0, 1] \setminus \{1/2\}$ ;
- (d)  $\mathcal{I}(\gamma_v(1)) < 0$ .

*Proof.* We consider the path  $\tilde{\gamma} : [0, +\infty) \rightarrow H^1(\mathbb{R}^2)$  given for each  $\tau \in [0, \infty)$  by

$$(\tilde{\gamma}(\tau))(x) := \begin{cases} \frac{\tau}{\tau_0} v\left(\frac{x}{\tau_0}\right) & \text{if } \tau \leq \tau_0, \\ v\left(\frac{x}{\tau}\right) & \text{if } \tau \geq \tau_0. \end{cases}$$

with  $\tau_0 \ll 1$  to be chosen later. The function  $\tilde{\gamma}$  is clearly continuous on the interval  $[0, +\infty)$  and in particular at its boundary 0. For  $\tau \geq \tau_0$ , Proposition 5.2 gives

$$\begin{aligned} \mathcal{I}(\tilde{\gamma}(\tau)) &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau^2}{2} \int_{\mathbb{R}^2} |v|^2 - \frac{\tau^{2+\alpha}}{2} \int_{\mathbb{R}^2} (I_\alpha * F(v)) F(v) \\ (10) \quad &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \left( \frac{\tau^2}{2} - \frac{\tau^{2+\alpha}}{2+\alpha} \right) \int_{\mathbb{R}^2} |v|^2, \end{aligned}$$

which attains its strict maximum in  $\tau = 1$  and is negative for  $\tau \geq \tau_1$ , for some  $\tau_1 \gg 1$ . For  $\tau \leq \tau_0$  we use (3) with  $\theta = (1 + \frac{\alpha}{2})\pi$  and then apply Proposition 2.1 to the function  $\tilde{\gamma}(\tau) / (\int_{\mathbb{R}^2} |\nabla \tilde{\gamma}(\tau)|^2)^{1/2}$ :

$$(11) \quad \int_{\mathbb{R}^2} |F(\tilde{\gamma}(\tau))|^{\frac{4}{2+\alpha}} \leq C \int_{\mathbb{R}^2} \min\{1, |\tilde{\gamma}(\tau)|^2\} e^{2\pi|\tilde{\gamma}(\tau)|^2} \leq C \frac{\int_{\mathbb{R}^2} |\tilde{\gamma}(\tau)|^2}{\int_{\mathbb{R}^2} |\nabla \tilde{\gamma}(\tau)|^2} = C \tau_0^2 \int_{\mathbb{R}^2} |v|^2,$$

therefore, because of the Pohožaev identity (Proposition 5.2) and the Hardy–Littlewood–Sobolev inequality (Proposition 2.2), we have

$$\begin{aligned} \mathcal{I}(\tilde{\gamma}(\tau)) &= \frac{\tau^2}{2\tau_0^2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau^2}{2} \int_{\mathbb{R}^2} |v|^2 - \int_{\mathbb{R}^2} (I_\alpha * F(\tilde{\gamma}(\tau))) F(\tilde{\gamma}(\tau)) \\ &\leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau^2}{2} \int_{\mathbb{R}^2} |v|^2 + C \left( \int_{\mathbb{R}^2} |F(\tilde{\gamma}(\tau))|^{\frac{4}{2+\alpha}} \right)^{1+\frac{\alpha}{2}}. \end{aligned}$$

Therefore, in view of (11) and the Pohožaev identity again, we deduce that

$$\begin{aligned}\mathcal{I}(\tilde{\gamma}(\tau)) &\leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{\tau_0^2}{2} \int_{\mathbb{R}^2} |v|^2 + C\tau_0^{2+\alpha} \left( \int_{\mathbb{R}^2} |v|^2 \right)^{1+\frac{\alpha}{2}} \\ &= \mathcal{I}(v) + \left( \frac{\tau_0^2}{2} - \frac{\alpha}{2(2+\alpha)} \right) \int_{\mathbb{R}^2} |v|^2 + C\tau_0^{2+\alpha} \left( \int_{\mathbb{R}^2} |v|^2 \right)^{1+\frac{\alpha}{2}},\end{aligned}$$

which is strictly less than  $\mathcal{I}(v)$  if  $\tau_0 = \tau_0(v)$  is chosen small enough. Therefore, the function  $\tilde{\gamma}$  verifies the following properties:

- (a')  $\tilde{\gamma}(0) = 0$ ;
- (b')  $\tilde{\gamma}(1) = v$ ;
- (c')  $\mathcal{I}(\tilde{\gamma}(\tau)) < \mathcal{I}(v)$  for every  $t \in [0, \tau_1] \setminus \{1\}$ ;
- (d')  $\mathcal{I}(\tilde{\gamma}(\tau_1)) < 0$ .

To get the required  $\gamma_v$  it suffices to take a suitable change of variable  $\gamma_v(t) := \tilde{\gamma}(T(\tau))$  for some function  $T \in C([0, 1], \mathbb{R})$  satisfying  $T(0) = 0$ ,  $T(1) = 1/2$  and  $T(\tau_1) = 1$ .  $\square$

We are now in position to prove the main theorem of this work.

*Proof of Theorem 1.1.* Let  $(u_n)_{n \in \mathbb{N}}$  be the Pohožaev–Palais–Smale sequence given by Proposition 3.1. Then, by Proposition 4.1, it converges weakly to a solution  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$  of  $(\mathcal{P})$ . By definition of groundstate,  $\mathcal{I}(u) \geq c$  and, by Proposition 5.2, we have  $\mathcal{P}(u) = 0$  (Proposition 5.2 is applicable in view of Proposition 5.1). Arguing as in [17, Theorem 1], we get successively

$$\begin{aligned}(12) \quad \mathcal{I}(u) &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\alpha}{2(2+\alpha)} \int_{\mathbb{R}^2} |u|^2 \\ &\leq \liminf_{n \rightarrow \infty} \left( \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{\alpha}{2(2+\alpha)} \int_{\mathbb{R}^2} |u_n|^2 \right) = \liminf_{n \rightarrow \infty} \left( \mathcal{I}(u_n) - \frac{\mathcal{P}(u_n)}{2+\alpha} \right) = b.\end{aligned}$$

If  $v \in H^1(\mathbb{R}^2) \setminus \{0\}$  is another solution of the Choquard equation  $(\mathcal{P})$ , we apply Proposition 5.3 to  $v$ :

$$\mathcal{I}(v) = \sup_{t \in [0, 1]} \mathcal{I}(\gamma_v(t)) \geq \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} \mathcal{I}(\gamma(t)) = b.$$

The solution  $v$  being arbitrary, by definition of groundstate one has  $b \leq c$ . Putting everything together, we get

$$c \leq \mathcal{I}(u) \leq b \leq c,$$

hence  $\mathcal{I}(u) = b = c$ . The proof is complete.  $\square$

We point out as a corollary of the proof of Theorem 1.1, that the convergence in Proposition 4.1 turns out to be actually a strong convergence in  $H^1(\mathbb{R}^2)$  and that this gives as a byproduct a compactness property of the set of groundstates of  $(\mathcal{P})$ .

**Corollary 5.4.** *Let  $(u_n)_{n \in \mathbb{N}}$  be a Pohožaev–Palais–Smale sequence satisfying the assumptions of Proposition 4.1 and in addition*

$$\lim_{n \rightarrow \infty} \mathcal{I}(u_n) = c.$$

Then, there exists  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$  solving  $(\mathcal{P})$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^2$  such that, up to subsequences,  $u_n(\cdot - x_n) \rightarrow_{n \rightarrow \infty} u$  strongly in  $H^1(\mathbb{R}^2)$ .

Moreover, the set of groundstates

$$\mathcal{S}_c := \{u \in H^1(\mathbb{R}^2); u \text{ solves } (\mathcal{P}) \text{ and } \mathcal{I}(u) = c\}$$

is compact, up to translations, in  $H^1(\mathbb{R}^2)$ .

*Proof.* We apply Proposition 4.1; the first alternative is excluded by our assumption and the continuity of the functional  $\mathcal{I}$  at 0. Therefore we get, up to translations,  $u_n \rightharpoonup u$  as  $n \rightarrow \infty$  in  $H^1(\mathbb{R}^2)$  and the function  $u \in H^1(\mathbb{R}^2) \setminus \{0\}$  solves  $(\mathcal{P})$ . As in the proof of Theorem 1.1, we get

(13)

$$\liminf_{n \rightarrow \infty} \left( \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \frac{\alpha}{2(2+\alpha)} \int_{\mathbb{R}^2} |u_n|^2 \right) \leq c = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \frac{\alpha}{2(2+\alpha)} \int_{\mathbb{R}^2} |u|^2,$$

from which it follows that  $u_n \rightarrow u$  strongly in  $H^1(\mathbb{R}^2)$  as  $n \rightarrow \infty$ .

To show the compactness of the set of groundstates  $\mathcal{S}_c$ , we consider an arbitrary sequence  $(u_n)_{n \in \mathbb{N}}$  in  $\mathcal{S}_c$ . Because of Proposition 5.2, it verifies  $\mathcal{P}(u_n) = 0$  for every  $n \in \mathbb{N}$ , so it satisfies the hypotheses of Proposition 4.1 and of the first part of the present corollary; therefore, up to subsequences and translations it will converge to some  $u$  which solves  $(\mathcal{P})$  and, by the continuity of the functional  $\mathcal{I}$  in  $H^1(\mathbb{R}^2)$ , we get  $u \in \mathcal{S}_c$ .  $\square$

We conclude this paper by the following result on additional qualitative properties of the solution  $u$ .

**Proposition 5.5.** *If  $F$  is even and nondecreasing on  $(0, \infty)$  and  $u$  is a groundstate solution of  $(\mathcal{P})$ , then  $u$  has constant sign and is radially symmetric with respect to some point  $a \in \mathbb{R}^N$ .*

*Proof.* The proof is the same as [17, Propositions 5.2 and 5.3]. We briefly sketch the argument for the convenience of the reader.

To prove the constant-sign property, consider the path  $\gamma_u$  defined in Proposition 5.3. Since  $F$  is an even function,  $\mathcal{I}(|v|) = \mathcal{I}(v)$  for every  $v \in H^1(\mathbb{R}^2)$ , hence  $\mathcal{I}(|\gamma_u(t)|) < \mathcal{I}(|\gamma_u(1/2)|) = b$  for every  $t \in [0, 1] \setminus \{1/2\}$ . From this, one easily deduces that the function  $|u|$  is a groundstate solution of  $(\mathcal{P})$ ; since  $F' \geq 0$ , we can apply the strong maximum principle and get  $|u| > 0$ , namely  $u$  has constant sign. Without loss of generality we assume now that  $u \geq 0$ .

For the symmetry, we follow the strategy of Bartsch, Weth and Willem [2] and its adaptation to the Choquard equation [16, 17]. For any closed half space  $H \subset \mathbb{R}^2$  we consider the reflection  $\sigma_H$  with respect to  $H$  and define, for every  $u \in H^1(\mathbb{R}^2)$ , the polarization (see for example [5])

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\} & \text{if } x \in H, \\ \min\{u(x), u(\sigma_H(x))\} & \text{if } x \notin H. \end{cases}$$

We first observe that [5, lemma 5.3]

$$\int_{\mathbb{R}^2} |\nabla u^H|^2 + |u^H|^2 = \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2$$

Moreover, since  $F$  is nondecreasing on  $(0, +\infty)$ , we have  $(F \circ u)^H = F \circ (u^H)$  and thus in view of the rearrangement inequality for the Riesz potential

$$\int_{\mathbb{R}^2} (I_\alpha * F(u^H)) F(u^H) = \int_{\mathbb{R}^2} (I_\alpha * F(u)^H) F(u)^H \leq \int_{\mathbb{R}^2} (I_\alpha * F(u)) F(u)$$

with equality if and only if either  $(F \circ u)^H = F \circ u$  or  $(F \circ u)^H = F \circ u \circ \sigma_H$  [16, lemma 5.3]. It follows thus that,  $\mathcal{I}(u^H) \leq \mathcal{I}(u)$ , with equality holding if and only if either  $F(u^H) = F(u)$  or  $F(u^H) = F(u \circ \sigma_H)$  on  $\mathbb{R}^2$ . From this and the definition of the level  $b$ , it follows that  $u^H$  is a ground state solutions of  $(\mathcal{P})$ , hence either  $F(u^H) = F(u)$  or  $F(u^H) = F(u \circ \sigma_H)$  on  $\mathbb{R}^2$ . In the former case we easily get  $f(u^H) = f(u)$ , hence  $u^H = u$ ; in the latter, we similarly get  $u^H = u \circ \sigma_H$ . The hyperplane  $H$  being arbitrary, in either case we conclude that the function  $u$  is radially symmetric with respect to some point  $a \in \mathbb{R}^2$  [16, lemma 5.4; 23, proposition 3.15].  $\square$

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